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Integrable systems on $so(4)$ related to XXX spin chains with boundaries

A V Tsiganov and O V Goremykin

St Petersburg State University, St Petersburg, Russia

E-mail: tsiganov@mph.phys.spbu.ru

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Abstract
We consider two-site XXX Heisenberg magnets with different boundary conditions, which are integrable systems on $so(4)$ possessing additional cubic and quartic integrals of motion. The separated variables for these models are constructed using the Sklyanin method.

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1. Introduction

In recent years much effort has been spent on the classification of integrable cases of multidimensional tops. The case of the four-dimensional top is particularly interesting since it has direct physical significance [1]. Some of the integrable systems discovered recently are new even in the classical setting; e.g. the equations of motion have the form of Euler–Poisson, Kirchhoff or Poincaré equations.

In this paper we will describe two new integrable cases of the four-dimensional top. Our construction is based on the so-called quadratic r-matrix algebras and on the use of integrable spin chains with different boundary conditions. Gaudin spin chains without boundaries have already been applied in the study of Euler, Lagrange, Neumann and Clebsch systems on $e(3)$ and Manakov and Steklov systems on $so(4)$ [2]. Some degenerate cases for the open Heisenberg spin chain are connected to the Goryachev–Chaplygin top [4], auxiliary symmetric Neumann system and Kowalevski–Goryachev–Chaplygin top [5]. The separated variables for all these models may be derived from the separation of variables for the XYZ spin chain [6].

Our examples are connected to XXX Heisenberg magnets with boundaries [7, 8]. The corresponding Lax matrices will be constructed using the Lax matrix for the standard two-site XXX Heisenberg magnet:

$$T(\lambda) = \begin{pmatrix} \lambda - s_3 + i\delta_1 & s_1 + is_2 \\ s_1 - is_2 & \lambda + s_3 + i\delta_1 \end{pmatrix} \begin{pmatrix} \lambda - t_3 + i\delta_2 & t_1 + it_2 \\ t_1 - it_2 & \lambda + t_3 + i\delta_2 \end{pmatrix}. \quad (1.1)$$

Here δ_i are numerical shifts of the spectral parameter λ. The dynamical variables s_i, t_i are coordinates on $so(4) = so(3) \oplus so(3)$ with the following Lie–Poisson brackets:

$$\{s_i, s_j\} = \varepsilon_{ijk} s_k \qquad \{s_i, t_j\} = 0 \qquad \{t_i, t_j\} = \varepsilon_{ijk} t_k \tag{1.2}$$

where ε_{ijk} is the totally skew-symmetric tensor.

The matrix $T(\lambda)$ (1.1) defines the representation of the Sklyanin algebra

$$\{\overset{1}{T}(\lambda), \overset{2}{T}(\nu)\} = [r(\lambda - \nu), \overset{1}{T}(\lambda)\overset{2}{T}(\nu)] \tag{1.3}$$

on generic symplectic leaves of $so(4)$. Here we use the standard notation $\overset{1}{T}(\lambda) = T(\lambda) \otimes I$, $\overset{2}{T}(\nu) = I \otimes T(\nu)$ and the r-matrix has the form

$$r(\lambda - \nu) = \frac{\mathrm{i}}{\lambda - \nu} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1.4}$$

Applying the standard machinery [7, 8] to $T(\lambda)$ (1.1), one gets another integrable system with cubic and quartic additional integrals of motion. In fact, the construction leads automatically to the separated variables.

2. Cubic integrals of motion

The main property of the Sklyanin algebra (1.3) is that for *any* numerical matrix $\mathcal{K}$, the coefficients of the trace of the matrix $\mathcal{K}T(\lambda)$ give rise the commutative subalgebra

$$\{\operatorname{tr} \mathcal{K}T(\lambda),\ \operatorname{tr} \mathcal{K}T(\nu)\} = 0.$$

All the generators of this subalgebra are linear polynomials on coefficients of entries $T_{ij}(\lambda)$, which are interpreted as integrals of motion for the integrable system associated with matrix $T(\lambda)$. For instance, representation (1.1) generates one linear and one quadratic integral of motion in variables s_i, t_i and the corresponding integrable system is equivalent to a special case of the Poincaré system [9].

According to [8], we can construct commutative subalgebras generated by *quadratic* polynomials on coefficients of $T_{ij}(\lambda)$. Let us introduce the matrix

$$\widetilde{T}(\lambda) = \mathcal{K}_d(\lambda) T(\lambda) \tag{2.5}$$

where $T(\lambda)$ is given by (1.1) and

$$\mathcal{K}_d(\lambda) = \begin{pmatrix} \lambda + \mathcal{A}_0 & a_1\lambda + a_0 \\ b_1\lambda + b_0 & 0 \end{pmatrix}. \tag{2.6}$$

Here a_k, b_k are arbitrary numerical parameters and $\mathcal{A}_0$ depends on the dynamical variables:

$$\mathcal{A}_0 = a_1(\mathrm{i}s_2 + \mathrm{i}t_2 - t_1 - s_1) - b_1(s_1 + t_1 + \mathrm{i}s_2 + \mathrm{i}t_2) - s_3 - t_3.$$

We can say that the dynamical matrix $\mathcal{K}_d$ (2.6) describes some *dynamical* boundary conditions.

The trace of $\widetilde{T}(\lambda)$,

$$\widetilde{\tau}(\lambda) = \operatorname{tr} \widetilde{T}(\lambda) = \lambda^3 + I_1\lambda + I_2$$

gives rise to the commutative subalgebra of the Sklyanin brackets (1.3). The polynomials $I_{1,2}$, quadratic and cubic in the variables s_i, t_i, are integrals of motion for some integrable system on $so(4)$.

To compare this system with the known examples of integrable systems on $so(4)$, we introduce two vectors $\mathbf{J} = (J_1, J_2, J_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ with entries

$$y_i = \varkappa(s_i - t_i) \qquad J_i = s_i + t_i \tag{2.7}$$

which satisfy to the following Lie–Poisson brackets:

$$\{J_i, J_j\} = \varepsilon_{ijk} J_k \qquad \{J_i, y_j\} = \varepsilon_{ijk} y_k \qquad \{y_i, y_j\} = \varkappa^2 \varepsilon_{ijk} J_k. \tag{2.8}$$

Because the physical quantities y_k, J_k should be real, the parameter $\varkappa^2$ must be real too and algebra (2.8) is reduced to its two real forms $so(4, \mathbb{R})$ and $so(3, 1, \mathbb{R})$ for positive and negative $\varkappa^2$ respectively. The corresponding Casimir elements are equal to

$$C_\varkappa = \varkappa^2 |\mathbf{J}|^2 + |\mathbf{y}|^2 \qquad C = (\mathbf{y}, \mathbf{J}). \tag{2.9}$$

Let $(\mathbf{y}, \mathbf{J})$ and $\mathbf{y} \times \mathbf{J}$ stand for the inner vector product and for the vector cross-product respectively. In variables (2.7) the Hamilton function is equal to I_1 up to constants:

$$\begin{aligned} H &= 2I_1 + \frac{C_\varkappa}{2\varkappa^2} + 2\delta_1\delta_2 \\ &= |\mathbf{J}|^2 + (\mathbf{a}, \mathbf{J})(\mathbf{b}, \mathbf{J}) + \varkappa^{-1}(\mathbf{b}, \mathbf{y} \times \mathbf{J}) - (\mathbf{b}, (\delta_1 - \delta_2)\varkappa^{-1}\,\mathbf{y} + (\delta_1 + \delta_2)\mathbf{J}) + 2(\mathbf{c}, \mathbf{J}) \end{aligned} \tag{2.10}$$

where the numerical vectors are

$$\mathbf{a} = (0, 0, 2\mathrm{i}) \qquad \mathbf{b} = (\mathrm{i}(a_1 + b_1), a_1 - b_1, \mathrm{i}) \qquad \mathbf{c} = (a_0 + b_0, -\mathrm{i}(a_0 - b_0), 0).$$

The additional integrals of motion K take the following form:

$$\begin{aligned} K = -4\mathrm{i}I_2 &= (\mathbf{b}, \mathbf{J})[2|\mathbf{J}|^2 + \varkappa^{-1}(\mathbf{a}, \mathbf{y} \times \mathbf{J} - (\delta_1 - \delta_2)\mathbf{y} + \varkappa(\delta_1 + \delta_2)\mathbf{J}) - \varkappa^{-2}C_\varkappa - 4\delta_1\delta_2] \\ &\quad - 2\varkappa^{-1}(\delta_1 - \delta_2)(\mathbf{c}, \mathbf{y}) + 2\varkappa^{-1}(\mathbf{c}, \mathbf{y} \times \mathbf{J}) + 2(\delta_1 + \delta_2)(\mathbf{c}, \mathbf{J}). \end{aligned} \tag{2.11}$$

The integrals of motion H and K are defined up to canonical transformations.

Suppose that the Hamiltonian function has to depend on the third component of the vector $\mathbf{y} \times \mathbf{J}$ only. The Hamiltonian (2.10) has such a form after rotation, $\mathbf{y} \to U\mathbf{y}$ and $\mathbf{J} \to U\mathbf{J}$, by the following Euler angles:

$$\phi = \frac{\pi}{2} - \mathrm{i}\frac{\ln a_1 - \ln b_1}{2} \qquad \psi = 0 \qquad \theta = \frac{\pi}{2} - \mathrm{i}\ln(-\mathrm{i} + 2\sqrt{a_1 b_1}) - \frac{\mathrm{i}}{2}\ln(4a_1 b_1 + 1).$$

This rotation acts on the numerical vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ in the following way:

$$\widetilde{\mathbf{a}} = U^{-1}\mathbf{a} = \left(0, 2\sqrt{1 - c^2}, 2\mathrm{i}c\right) \qquad \widetilde{\mathbf{b}} = U^{-1}\mathbf{b} = (0, 0, -\mathrm{i}c^{-1})$$

$$\widetilde{\mathbf{c}} = U^{-1}\mathbf{c} = \frac{1}{2c}\left(\alpha, \beta, -\frac{\mathrm{i}\sqrt{1 - c^2}}{c}\beta\right)$$

where

$$c = \frac{1}{\sqrt{4a_1 b_1 + 1}} \qquad \alpha = \frac{2\mathrm{i}(a_1 b_0 - a_0 b_1)}{\sqrt{a_1 b_1 (4a_1 b_1 + 1)}} \qquad \beta = \frac{-2(a_1 b_0 + a_0 b_1)}{\sqrt{a_1 b_1 (4a_1 b_1 + 1)}}.$$

Substituting $\widetilde{\mathbf{a}}$, $\widetilde{\mathbf{b}}$ and $\widetilde{\mathbf{c}}$ instead of vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ into (2.10) and (2.11), one gets integrals of motion after rotation. The Hamilton function H (2.10) after rotation and renormalization,

$$\begin{aligned} \widehat{H} = \frac{H}{\sqrt{4a_1 b_1 + 1}} &= c\left(J_1^2 + J_2^2 - J_3^2\right) - 2\sqrt{1 - c^2}\,J_2 J_3 \\ &\quad + \frac{1}{\mathrm{i}\varkappa}(y_2 J_1 - y_1 J_2) + \alpha J_1 + \beta J_2 + \gamma J_3 + \delta y_3 \end{aligned} \tag{2.12}$$

depends on five essential parameters: c, α, β,

$$\gamma = -\mathrm{i}(\delta_1 + \delta_2) + 4\frac{a_1 b_0 + a_0 b_1}{4a_1 b_1 + 1} \qquad \delta = \frac{\delta_2 - \delta_1}{\mathrm{i}\varkappa}.$$

It is a real function on $so(3, 1, \mathbb{R})$ with negative $\varkappa^2$ only.

According to [6, 8], the separated coordinates $q_{1,2}$ for (2.12) are zeros of the polynomial

$$T_{11}(\lambda) = (\lambda - q_1)(\lambda - q_2) = 0 \tag{2.13}$$

whereas the conjugated momenta are equal to

$$p_k = -\mathrm{i} \ln T_{21}(q_k) - \ln(a_1 q_k + a_0). \tag{2.14}$$

We can prove that q_k, p_k are Darboux variables using (2.13) and (2.14) and brackets (1.3).

By definition, the generating function of the integrals of motion is

$$\begin{aligned}\widetilde{\tau}(\lambda) &= \operatorname{trace} \widetilde{T}(\lambda) = \lambda^3 + I_1\lambda - I_2 \\ &= (\lambda + \mathcal{A}_0)T_{11}(\lambda) + (a_1\lambda + a_0)T_{21}(\lambda) + (b_1\lambda + b_0)T_{12}(\lambda).\end{aligned}$$

Substituting $\lambda = q_k$ into this equation one gets two separated equations:

$$q_k^3 + I_1 q_k + I_2 = \exp(\mathrm{i}p_k) + \det \widetilde{T}(q_k) \exp(-\mathrm{i}p_k) \qquad k = 1, 2.$$

Here we took into account that $T_{11}(q_k) = 0$ and $T_{12}(q_k) = \det T(q_k) T_{21}^{-1}(q_k)$.

3. Quartic integrals of motion

According to [7], we can construct another commutative subalgebra generated by *quadratic* polynomials on coefficients of $T_{ij}(\lambda)$, which are integrals of motion for another integrable system associated with the same matrix $T(\lambda)$ (1.1). We recall that if $\mathcal{K}_\pm(\lambda)$ are solutions of the reflection equation in classical mechanics:

$$\{\overset{1}{\mathcal{T}}(\lambda), \overset{2}{\mathcal{T}}(\nu)\} = [r(\lambda - \nu), \overset{1}{\mathcal{T}}(\lambda)\overset{2}{\mathcal{T}}(\nu)] + \overset{1}{\mathcal{T}}(\lambda) r(\lambda + \nu) \overset{2}{\mathcal{T}}(\nu) - \overset{2}{\mathcal{T}}(\nu) r(\lambda + \nu) \overset{1}{\mathcal{T}}(\lambda) \tag{3.15}$$

then the coefficients of the trace of the Lax matrix,

$$L(\lambda) = \mathcal{K}_-(\lambda) T(\lambda) \mathcal{K}_+(\lambda) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} T^t(-\lambda) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{3.16}$$

give rise to the commutative subalgebra

$$\{\operatorname{tr} L(\lambda),\ \operatorname{tr} L(\nu)\} = 0.$$

In (3.16) the superscript t stands for matrix transposition; matrix $T(\lambda)$ satisfies (1.3) and commutes with $\mathcal{K}_\pm(\lambda)$.

For the rational r-matrix (1.4), the numerical solutions of the classical equation (3.15) coincide with the solutions of the quantum reflection equation (see [7]), which were found in [10]. We shall use the following parametrization of these solutions:

$$\mathcal{K}_+ = \begin{pmatrix} b_3\lambda + \alpha & (b_1 + \mathrm{i}b_2)\lambda \\ (b_1 - \mathrm{i}b_2)\lambda & -b_3\lambda + \alpha \end{pmatrix} \qquad \mathcal{K}_- = \begin{pmatrix} a_3\lambda + \beta & (a_1 + \mathrm{i}a_2)\lambda \\ (a_1 - \mathrm{i}a_2)\lambda & -a_3\lambda + \beta \end{pmatrix}. \tag{3.17}$$

Inserting matrix $T(\lambda)$ (1.1) and these boundary matrices $\mathcal{K}_\pm$ into (3.16), one gets the Lax matrix $L(\lambda)$ for the two-site XXX Heisenberg magnet with boundaries. The trace of $L(\lambda)$,

$$\tau(\lambda) = \operatorname{tr} L(\lambda) = -2(\mathbf{a}, \mathbf{b})\lambda^6 - I_1\lambda^4 - I_2\lambda^2 - I_3 \tag{3.18}$$

gives rise to the commutative subalgebra of the Sklyanin brackets (1.3). The integrals of motion I_1, I_2 and I_3 are second-, fourth- and sixth-order polynomials in the variables s_i, t_i.

In variables (2.7), the Hamilton function H (3.19) is equal to I_1 up to constants:

$$\begin{aligned} H &= I_1 + (\mathbf{a}, \mathbf{b})\big(\varkappa^{-2} C_\varkappa - 2\big(\delta_1^2 + \delta_2^2\big)\big) - 2\alpha\beta \\ &= (\mathbf{J}, A\mathbf{J}) + \varkappa^{-1}(\mathbf{a} \times \mathbf{b}, \mathbf{y} \times \mathbf{J}) + 2(\alpha\mathbf{a} + \beta\mathbf{b}, J) \\ &\quad - \varkappa^{-1}(\delta_1 - \delta_2)(\mathbf{a} \times \mathbf{b}, \mathbf{y}) - (\delta_1 + \delta_2)(\mathbf{a} \times \mathbf{b}, \mathbf{J}) \end{aligned} \tag{3.19}$$

where $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{b} = (b_1, b_2, b_3)$ are numerical vectors and

$$A = \mathbf{a} \otimes \mathbf{b} + \mathbf{b} \otimes \mathbf{a} \qquad A_{ij} = a_i b_j + a_j b_i. \tag{3.20}$$

The additional integral of motion

$$K = 2\varkappa^2 I_2 - (\mathbf{a}, \mathbf{b}) \left(C^2 + \frac{C_\varkappa^2}{4\varkappa^2} \right) + 2\alpha\beta C_\varkappa \tag{3.21}$$

is a third-order polynomial in momenta $\mathbf{J}$. For brevity, we present K at $\delta_1 = \delta_2 = 0$ only:

$$\begin{aligned} K = (\varkappa^2 |\mathbf{J}|^2 - |\mathbf{y}|^2)[\varkappa^{-1}(\mathbf{a} \times \mathbf{b}, \mathbf{y} \times \mathbf{J}) + 2(\alpha\mathbf{a} + \beta\mathbf{b}, J)] + |\mathbf{J}|^2 [8\varkappa^2\alpha\beta - 2(\mathbf{a}, \mathbf{b})|\mathbf{y}|^2 \\ + 4\varkappa(\alpha\mathbf{a} - \beta\mathbf{b}, \mathbf{y})] + (\mathbf{y} \times \mathbf{J}, A\mathbf{y} \times \mathbf{J}) - 4\varkappa(\mathbf{y}, \mathbf{J})(\alpha\mathbf{a} - \beta\mathbf{b}, \mathbf{J}). \end{aligned} \tag{3.22}$$

The third coefficient I_3 is a constant:

$$I_3 = \left(\frac{C_\varkappa}{4\varkappa^2} + \frac{C}{2\varkappa} + \delta_1^2 \right) \left(\frac{C_\varkappa}{4\varkappa^2} - \frac{C}{2\varkappa} + \delta_2^2 \right). \tag{3.23}$$

The integrals of motion (3.19) and (3.22) depend on ten numerical parameters $a_i, b_i, \alpha, \beta, \delta_1, \delta_2$ and they are defined up to canonical transformations.

Suppose that the Hamiltonian function has to depend on the third component of the vector $\mathbf{y} \times \mathbf{J}$ only. The Hamiltonian (3.19) has such a form after the rotation $\mathbf{J} \to U\mathbf{J}$ and $\mathbf{y} \to U\mathbf{y}$, with the orthogonal matrix U defined by

$$\widetilde{\mathbf{a}} = U^{-1}\mathbf{a} = \left(\sqrt{\frac{e_1}{2}}, \mathrm{i}\sqrt{\frac{e_2}{2}}, 0 \right) \qquad \widetilde{\mathbf{b}} = U^{-1}\mathbf{b} = \left(\sqrt{\frac{e_1}{2}}, -\mathrm{i}\sqrt{\frac{e_2}{2}}, 0 \right) \tag{3.24}$$

where e_i are eigenvalues of the matrix A (3.20). Substituting $\widetilde{\mathbf{a}}, \widetilde{\mathbf{b}}$ instead of vectors $\mathbf{a}, \mathbf{b}$ into (3.19) and (3.22), one gets the integrals of motion after rotation.

If $\mathbf{a}$ and $\mathbf{b}$ are linearly dependent vectors, then $\mathbf{a} \times \mathbf{b} = 0$ and matrix A (3.20) has only one non-zero value: $e_1 \simeq |\mathbf{a}|^2$. In this case the Hamilton function (3.19)

$$\widetilde{H} = J_1^2 + cJ_1 \qquad c \in \mathbb{R}$$

determines the degenerate or superintegrable system with a noncommutative family of additional integrals of motion. For instance, there is the following quadratic integral:

$$\widetilde{K} = c_1 J_1^2 + J_2^2 + J_3^2 + c_2 y_1^2 + c_3 \big(y_2^2 + y_3^2\big) + c_4 y_1 J_1 + c_5 \big(y_2 J_3 - y_3 J_2\big).$$

This is a special case of the Poincaré model [9]. For more details, see [11].

If $\mathbf{a} \times \mathbf{b} \neq 0$, the matrix A (3.20) has two non-zero eigenvalues:

$$e_{1,2} = (\mathbf{a}, \mathbf{b}) \pm |\mathbf{a}||\mathbf{b}| \qquad e_3 = 0.$$

In this case, after rotation and renormalization the Hamiltonian H (3.19) is equal to

$$\begin{aligned} \widetilde{H} = \frac{H}{\mathrm{i}\sqrt{e_1 e_2}} &= cJ_1^2 - c^{-1}J_2^2 + \varkappa^{-1}(y_1 J_2 - y_2 J_1) \\ &\quad + \widetilde{\alpha} J_1 + \widetilde{\beta} J_2 - (\delta_1 + \delta_2) J_3 - \varkappa^{-1}(\delta_1 - \delta_2) y_3 \end{aligned} \tag{3.25}$$

where

$$c = -\mathrm{i}\sqrt{e_1 e_2^{-1}} \qquad \widetilde{\alpha} = c_1 = -\mathrm{i}\sqrt{2e_2^{-1}}(\alpha + \beta) \qquad \widetilde{\beta} = \sqrt{2e_1^{-1}}(\alpha - \beta).$$

The Hamilton function $\widetilde{H}$ (3.25) depends on five parameters instead of ten parameters as for the initial Hamiltonian (3.19). This allows us to impose constraints on the vectors $\mathbf{a}$ and $\mathbf{b}$ and to use triangular boundary matrices $\mathcal{K}_\pm$ [7] instead of the general ones (3.17). Similar points hold for the BC_n Toda lattices [8] and for the Kowalevski–Goryachev–Chaplygin gyrostats [12], which are also related to the reflection equations.

Thus we can consider a low triangular solution of the reflection equations $\mathcal{K}_+$ (3.17),

$$b_1 - \mathrm{i}b_2 = 0 \tag{3.26}$$

without loss of generality. According to the Sklyanin method [6] the separated variables may be defined by entries of the following intermediate matrix:

$$\mathcal{T}(\lambda) = T(\lambda)\mathcal{K}_-(\lambda)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} T^t(-\lambda)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{3.27}$$

which satisfies to the classical reflection equation (3.15).

The separated coordinates $q_{1,2}$ are non-trivial zeros of the polynomial

$$\mathcal{T}_{12}(\lambda) = \lambda\big(\lambda^2 - q_1^2\big)\big(\lambda^2 - q_2^2\big) = 0 \tag{3.28}$$

whereas the conjugated momenta are equal to

$$p_k = -\mathrm{i}\ln\mathcal{T}_{11}(q_k) - \ln(b_3 q_k + \alpha). \tag{3.29}$$

We can prove that q_k, p_k are Darboux variables using (3.28) and (3.29) and the reflection equation (3.15).

As $\mathcal{K}_+$ is a low triangular matrix, the generating function of the integrals is

$$\begin{aligned}\tau(\lambda) &= \operatorname{trace} L(\lambda) = -2(\mathbf{a},\mathbf{b})\lambda^6 - I_1\lambda^4 - I_2\lambda^2 - I_3 \\ &= \operatorname{trace}\mathcal{K}_+\mathcal{T}(\lambda) = (b_3\lambda+\alpha)\mathcal{T}_{11}(\lambda) + (b_1+\mathrm{i}b_2)\lambda\mathcal{T}_{12}(\lambda) - (b_3\lambda-\alpha)\mathcal{T}_{22}(\lambda).\end{aligned}$$

Substituting $\lambda = q_k$ into this equation, one gets two separated equations:

$$2(\mathbf{a},\mathbf{b})q_k^6 + I_1 q_k^4 + I_2 q_k^2 + I_3 = \exp(\mathrm{i}p_k) + \det L(q_k)\exp(-\mathrm{i}p_k) \qquad k = 1, 2.$$

Here we took into account that $\mathcal{T}_{12}(q_k) = 0$ and $\mathcal{T}_{22}(q_k) = \det\mathcal{T}(q_k)\mathcal{T}_{11}^{-1}(q_k)$.

According to [12] we can introduce other separated variables related to the proposed separated variables by canonical transformation and by flips of parameters. The existence of the different separated variables is associated with the invariance of the Sklyanin brackets with respect to a matrix transposition $T \to T^t$.

4. Conclusion

The Hamiltonians (2.12) and (3.19) belong to the following class of the Hamiltonians:

$$H = (\mathbf{J}, A\mathbf{J}) + (\mathbf{a}, \mathbf{y}\times\mathbf{J}) + (\mathbf{b}, \mathbf{J}) + (\mathbf{c}, \mathbf{y}) \tag{4.30}$$

possessing additional integrals of third and fourth degree in momenta. V V Sokolov kindly informed us that there exist three different such integrable cases only [13].

The list of interesting integrable tops equally covered by the general scheme may be extended. For instance, similar three- and four-site Heisenberg magnets with boundaries describe systems of three and four interacting tops in $\mathbb{R}^3$, which presumably have a physical meaning in connection with the motion of a rigid body with elliptic cavities filled with ideal fluid. Another way to proceed is to consider another representation $T(\lambda)$ of the Sklyanin algebra in 3×3 auxiliary matrix space.

On the other hand, starting with the quantum r-matrix algebras (1.3), (3.15) and using the quantum operator $T(\lambda)$ (1.1) we can construct the quantum counterpart of the proposed systems. Furthermore, the standard technique of the quantum inverse scattering method also gives solutions in the quantum case.

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